Editorial - Magic City

Problem summary

Given a positive integer K ($1 \le K \le 50$), we need to construct an undirected graph with $N \le 2000$ vertices such that:

- Each vertex is assigned a color (attraction type) from 1 to $2K$. Every color is used at least once.
- For every ordered triple of colors (i, j, k) with $1 \le i, j, k \le 2K$, $i \neq j$, and $j \neq k$ (note that $i = k$ is allowed), there exists a walk of length 2 whose vertices have colors i, j, k in this order.
- Every vertex has degree at most K.

The goal is to minimize N. For $K = 1, 2, 3, 4, 5$, full score requires $N \le 2, 12, 24, 40, 50$, respectively; for $6 \le K \le 50$, full score requires $N \le 12K$.

Basic $O(K^3)$ Solutions (17.68 points)

One idea is that, for each tuple (i, j, k), we create a separate 3-vertex path with colors $i - j - k$ as an isolated component. Since there are $2K(2K-1)^2$ such triples, this uses $N = 6K(2K-1)^2$ vertices.

To reduce N, we can instead create a triangle (K_3) with colors i, j, k for each unordered triple $1 \le i < j < k \le 2K$. A single triangle covers all 6 permutations of (i, j, k) as well as triples with repeating colors, such as (i, j, i) and (j, i, j). Every vertex has degree $2 \le K$ (for $K \ge 2$).

This uses $N = 3\binom{2K}{3} = K(2K-1)(2K-2)$ vertices, which gives about 17.68 points.

Solution with $64K$ Vertices (24.73 points)

To reduce the number of vertices to $O(K)$, we need components of $\Theta(K)$ vertices that each cover $\Theta(K^3)$ triples. Consider a component partitioned into three independent sets X, Y, Z, with complete bipartite graphs between X and Y and between Y and Z. Each of X, Y, Z corresponds to a group of colors and contains exactly one vertex of each color in its group. Then every triple (i, j, k) with i in the color group of X, j in that of Y, and k in that of Z is covered by a walk $X - Y - Z$.

To satisfy the degree constraint $\le K$, we need $|Y| \le K$ and $|X| + |Z| \le K$. When K is even, we can simply set $|X| = |Z| = \frac{1}{2}K$ and $|Y| = K$. Partition the $2K$ colors into 4 groups of size $\frac{1}{2}K$ for X and Z, and into 2 groups of size K for Y. Creating one component for each of the $4 \times 2 \times 4 = 32$ combinations of groups covers all triples. This uses $32 \times 2K = 64K$ vertices.

The case when K is odd requires more care. It is worth noting that $\lceil K/2 \rceil + \lceil K/2 \rceil = K + 1 > K$, violating the degree bound for Y. We can instead split the colors for X into 5 groups of size at most $\lceil \frac{2}{5}K \rceil$ and for Z into 4 groups of size at most $\lceil \frac{1}{2}K \rceil$. The degree of a Y vertex is then $\lceil \frac{2}{5}K \rceil + \lceil \frac{1}{2}K \rceil \approx \frac{9}{10}K$, which is at most K for every odd $K \ge 5$ (but not for $K = 3$), and the $5 \times 2 \times 4 = 40$ components use $\frac{19}{10}K \times 40 = 76K$ vertices. More generally, one can choose group sizes a and c with $a + c = K$ minimizing $\lceil 2K/a \rceil \cdot 2 \cdot \lceil 2K/c \rceil$.

On its own, this solution gets about 15.94 points. Combining with the $O(K^3)$ solution, we can achieve 24.73 points.

Further optimization ($24K$ for even, $36K$ for odd) (40.21 points)

Let's try to optimize the even case first. Consider the following two observations:

1. Embedding the walk $i-j-k$ also embeds $k-j-i$. If we partition the $2K$ colors into four groups A, B, C, D of size $\frac{1}{2}K$, we only need the 10 unordered pairs of groups $(A,A),(A,B),(A,C),(A,D),(B,B),(B,C),(B,D),(C,C),(C,D),(D,D)$ for (X,Z), each paired with the 2 groups of size K for Y. This reduces the number of components from 32 to 20, using $40K$ vertices.
2. The pairs $(A,A),(B,B),(C,C),(D,D)$ are redundant: any walk $i-j-k$ with $i,k \in A$ and $j \in Y$ is already embedded as a path through $X-Y-X$ inside the (A,B) component. Thus, only the $\binom{4}{2}=6$ pairs of distinct groups are needed for (X,Z), requiring $6 \times 2 = 12$ components and $N = 24K$ vertices.

For odd K, partitioning the colors into 5 groups in the same manner requires $\binom{5}{2} \times 2 = 20$ components of size $\frac{9}{5}K$, giving $N = 36K$ vertices. Here a Y vertex has degree at most $2\lceil \frac{2}{5}K \rceil$, which is at most K for every odd $K \geq 5$; the case $K = 3$ is handled by the $O(K^3)$ solution.

Solution with $20K$ Vertices (51.26 points)

Another approach is to use disjoint complete graphs. A clique whose vertices have color set S covers all valid triples (i,j,k) with $i,j,k \in S$, and satisfies the degree bound whenever $|S| \leq K+1$.

Partition the $2K$ colors as evenly as possible into 6 groups A, B, C, D, E, F of size $\approx \frac{1}{3}K$. For each of the $\binom{6}{3} = 20$ triples of groups, form a complete graph on their union. Since any three colors belong to at most 3 groups, all triples are covered. Each color belongs to $\binom{5}{2} = 10$ cliques, giving $N = 20K$ vertices.

This construction works for all K, even when K is not divisible by 3:

- If $K = 3n+1$, the group sizes are $n+1, n+1, n, n, n, n$, so the largest clique has $3n+2 = K+1$ vertices.
- If $K = 3n+2$, the group sizes are $n+1, n+1, n+1, n+1, n, n$, so the largest clique has $3n+3 = K+1$ vertices.

In both cases, $|S| \leq K+1$ holds for every component.

Solution with $14K$ Vertices (65.32 points)

Instead of 6 groups, partition the $2K$ colors into 8 groups A, B, C, D, E, F, G, H of size $\approx \frac{1}{4}K$, and construct 14 cliques corresponding to the blocks of the Steiner quadruple system $S(3,4,8)$:

$$(A,B,C,D),\ (A,B,E,F),\ (A,B,G,H),\ (A,C,E,G),\ (A,C,F,H),\ (A,D,E,H),\ (A,D,F,G),$$

$$(E,F,G,H),\ (C,D,G,H),\ (C,D,E,F),\ (B,D,F,H),\ (B,D,E,G),\ (B,C,F,G),\ (B,C,E,H).$$

Every triple of groups is contained in one of these 14 quadruples, and each group appears in 7 cliques, yielding $N = 14K$ vertices.

We must be careful when $K \equiv 2 \pmod 4$. Write $K = 4n+2$; then four of the eight groups have size $n+1$ and the other four have size n. If the four larger groups formed a block (for example, A, B, C, D), that clique

would have size $4n + 4 = K + 2 > K + 1$, violating the degree constraint. Instead, choose the four larger groups so that they do **not** form a block, for example A, B, C, E. Since every triple of groups lies in exactly one block, no block contains all four larger groups, so the largest clique has size $3(n + 1) + n = K + 1$. For other residues of K modulo 4, every clique has size at most $K + 1$ regardless of the arrangement.

Hence the $14K$ construction works for all K. Combined with the $O(K^3)$ solution for small K, it gets 65.32 points.

Reformulation via the Middle Vertex

Consider the problem from the perspective of the middle color c in each walk $x - c - y$. The walk condition is equivalent to requiring the following property for every color $c \in \{1, \ldots, 2K\}$:

> $(\star)$ Let $v_1, \ldots, v_k$ be the vertices of color c, and let $S_i \subseteq \{1, \ldots, 2K\} \setminus \{c\}$ be the set of colors adjacent to v_i ($|S_i| \leq K$). For every $x, y \in \{1, \ldots, 2K\} \setminus \{c\}$, there exists $i \in \{1, \ldots, k\}$ such that $\{x, y\} \subseteq S_i$.

Because $U_c = \{1, \ldots, 2K\} \setminus \{c\}$ has size $2K - 1$ rather than $2K$, subproblem $(\star)$ always admits a solution with $k = 6$ sets regardless of the parity of K:

- Partition U_c into four groups S_A, S_B, S_C, S_D as evenly as possible, and set

$$(S_1, S_2, S_3, S_4, S_5, S_6) = (S_A \cup S_B,\ S_A \cup S_C,\ S_A \cup S_D,\ S_B \cup S_C,\ S_B \cup S_D,\ S_C \cup S_D).$$

- If $K = 2n$, the group sizes are $n, n, n, n - 1$, so $\max_i |S_i| = 2n = K$.
- If $K = 2n + 1$, the group sizes are $n + 1, n, n, n$, so $\max_i |S_i| = 2n + 1 = K$.

Using this $k = 6$ construction, we can achieve $N = 24K$ for all K via a bipartite graph between an **upper part** (vertices acting as middle points of length-2 walks) and a **lower part**:

- Place 6 vertices of each color $c \in \{1, \ldots, 2K\}$ in the upper part ($12K$ vertices in total), and assign them the neighbor-color sets $S_1, \ldots, S_6$ from $(\star)$.
- In our construction for $(\star)$, each color $x \in U_c$ appears in exactly 3 of the 6 sets $S_1, \ldots, S_6$. Summing over all $2K$ colors c, each color x is required $3(2K - 1) = 6K - 3 \leq 6K$ times by upper-part vertices.
- Place 6 vertices of each color x in the lower part ($12K$ vertices in total). Since each lower-part vertex can accept up to K edges, 6 vertices of color x have total capacity $6K \geq 6K - 3$ and can be connected to all upper-part vertices requiring color x.

Solution with $12K$ Vertices (Optimal for $K \geq 6$)

As shown in the Appendix, $(\star)$ has no solution with $k = 5$ for $K \geq 6$, so every color needs at least 6 vertices, and every valid graph has $N \geq 12K$. To reach exactly $12K$ vertices, every color must have exactly 6 vertices, and every vertex must act as a middle point, so the lower part of the previous construction must disappear.

Create 6 vertices for each color $c \in \{1, \ldots, 2K\}$ ($N = 12K$ vertices in total), and assign them the 6 desired neighbor-color sets $S_1, \ldots, S_6$ from the $k = 6$ solution to $(\star)$.

For a vertex u of color c_1 and a color $c_2 \in S_u$, view this requirement as a directed half-edge ("arrow") $c_1 \to c_2$ incident to u. We can add an undirected edge between a vertex u of color c_1 and a vertex v of color c_2 by pairing a $c_1 \to c_2$ arrow at u with a $c_2 \to c_1$ arrow at v.

Since each $c_2 \neq c_1$ appears in exactly 3 of the 6 sets for c_1, every pair of distinct colors $\{c_1, c_2\}$ has exactly

three $c_1 \to c_2$ arrows and three $c_2 \to c_1$ arrows. Pairing them arbitrarily produces a valid graph with $N = 12K$ vertices.

Optimal Solutions for $K \leq 5$

There are two ways to find the optimal solutions for smaller cases. First, we will introduce the method by hand. Let $f(K)$ be the minimum number of sets k required in subproblem $(\star)$ on a universe of size $2K-1$ with $|S_i| \leq K$. Since every color must appear at least $f(K)$ times, any valid graph has at least $2K \cdot f(K)$ vertices. We have $f(K) = 6$ for $K \geq 6$, while for $K \leq 5$ the optimal values and set families (on $\{1, \ldots, 2K-1\}$) are:

- $K = 1,\ f(1) = 1$: $S_1 = \{1\}$.
- $K = 2,\ f(2) = 3$: $S_1 = \{1, 2\},\ S_2 = \{1, 3\},\ S_3 = \{2, 3\}$.
- $K = 3,\ f(3) = 4$: $S_1 = \{1, 2, 3\},\ S_2 = \{1, 4, 5\},\ S_3 = \{2, 4, 5\},\ S_4 = \{3, 4, 5\}$.
- $K = 4,\ f(4) = 5$:
  $S_1 = \{1, 2, 3, 4\},\ S_2 = \{1, 5, 6, 7\},\ S_3 = \{2, 5, 6, 7\},\ S_4 = \{3, 5, 6, 7\},\ S_5 = \{4, 5, 6, 7\}$.
- $K = 5,\ f(5) = 5$:
  $S_1 = \{1, 2, 3, 4, 5\},\ S_2 = \{1, 6, 7, 8, 9\},\ S_3 = \{2, 6, 7, 8, 9\},\ S_4 = \{3, 4, 5, 6, 7\},\ S_5 = \{3, 4, 5, 8, 9\}$
  .

For $K = 1$ and $K = 2$, every element in $\{1, \ldots, 2K-1\}$ appears the same number of times (1 and 2 times, respectively) across $S_1, \ldots, S_{f(K)}$, so all $c_1 \to c_2$ and $c_2 \to c_1$ arrows can be matched directly.

For $K = 3, 4, 5$, elements of $\{1, \ldots, 2K-1\}$ have different frequencies in $S_1, \ldots, S_{f(K)}$. To ensure that the number of $c_1 \to c_2$ arrows equals the number of $c_2 \to c_1$ arrows for every pair $\{c_1, c_2\}$, we must assign a permutation of $\{1, \ldots, 2K-1\}$ to $\{1, \ldots, 2K\} \setminus \{c\}$ for each color c. Equivalently, we need an edge-weighting of K_{2K} (where the weight of edge (c_1, c_2) is the number of arrows between c_1 and c_2) such that at every vertex, the multiset of incident edge weights matches the element frequencies in $S_1, \ldots, S_{f(K)}$:

- For $K = 3$, the element frequencies are $2, 2, 2, 3, 3$. Assign weight 3 to the edges of a 6-cycle and weight 2 to the remaining edges of K_6.
- For $K = 4$, the element frequencies are $2, 2, 2, 2, 4, 4, 4$. Partition K_8 into a 3-regular graph of weight 4 and a 4-regular graph of weight 2.
- For $K = 5$, the element frequencies are $2, 2, 3, 3, 3, 3, 3, 3, 3$. Assign weight 2 to the edges of a 10-cycle and weight 3 to the remaining edges of K_{10}.

Concretely, this weighting can be realized uniformly by identifying the elements of $\{1, \ldots, 2K-1\}$ with cyclic shifts $\Delta \in \{1, \ldots, 2K-1\} \pmod{2K}$ (so a vertex of color c has arrows to $(c + \Delta) \bmod 2K$) and relabeling the universe so that Δ and $2K - \Delta$ always have the same frequency in $S_1, \ldots, S_{f(K)}$:

- $K = 3$ ($2K = 6$): map the two frequency-3 elements to $\{1, 5\}$ and the three frequency-2 elements to $\{2, 3, 4\}$.
- $K = 4$ ($2K = 8$): map the four frequency-2 elements to $\{1, 7, 2, 6\}$ and the three frequency-4 elements to $\{3, 5, 4\}$.
- $K = 5$ ($2K = 10$): map the two frequency-2 elements to $\{1, 9\}$ and the seven frequency-3 elements to $\{2, 8, 3, 7, 4, 6, 5\}$.

This ensures that every pair of colors has the exact same number of arrows in both directions. Matching them yields $N = 2, 12, 24, 40, 50$ vertices for $K = 1, 2, 3, 4, 5$, achieving 100 points.

Finding these by brute force. On the other hand, since the instances are tiny, a simple backtracking search finds them directly. Elements are the shifts $\Delta \in \{1, \ldots, 2K-1\}$, and we search for m sets of size exactly K

(enlarging a set never hurts the covering). At each step, take the smallest uncovered pair $\{x, y\}$ (including $x = y$) and branch only on sets containing it, choosing sets in increasing order. At the leaves, additionally check that Δ and $2K - \Delta$ have the same frequency. For every $K \le 5$, this search runs in well under a second. It finds a valid family for $m = 1, 3, 4, 5, 5$. It also proves that no family exists with one fewer set, even without the symmetry check, so it confirms $f(K)$ for $K \le 5$ without the case analysis below. The arrows are then paired exactly as in the $12K$ solution.

Appendix: Proof of the Lower Bound $f(K)$

We now prove that $f(1) = 1$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$, $f(5) = 5$, and $f(K) = 6$ for all $K \ge 6$. The upper bounds come from the constructions above, and $f(1) = 1$ is trivial, so it remains to prove the lower bounds. Assume $K \ge 2$, and let $U = \{1, \dots, 2K - 1\}$.

Dual view. For each $j \in U$, define $T_j = \{i \in \{1, \dots, k\} : j \in S_i\}$. The condition that every pair $\{x, y\} \subseteq U$ (possibly $x = y$) is contained in some S_i is equivalent to $T_x \cap T_y \neq \emptyset$ for all $x, y \in U$. The sets T_j need not be distinct; a solution is determined by how many elements $j \in U$ have $T_j = T$, for each $T \subseteq \{1, \dots, k\}$. Dividing these counts by $|U|$, we obtain shares $w_T \ge 0$ with $\sum_T w_T = 1$, and the *load* of index i is $|S_i|/|U| = \sum_{T \ni i} w_T$. We **relax** the problem by allowing arbitrary real shares: any lower bound on $\max_i |S_i|/|U|$ for the relaxed problem is also a lower bound for the original problem.

Simplifications. Let $\mathcal{T}$ be the family of sets T with $w_T > 0$; it is *intersecting* (any two members intersect).

- $\bigcap_{T \in \mathcal{T}} T = \emptyset$: otherwise some i lies in every T_j, giving $|S_i| = 2K - 1 > K$.
- Every $T \in \mathcal{T}$ has $|T| \ge 2$: if $\{c\} \in \mathcal{T}$, then every other member must intersect it, so c lies in all of them, contradicting the previous point.
- We may assume $\mathcal{T}$ is an *antichain*: if $T \subsetneq T'$ are both in $\mathcal{T}$, moving the share of T' to T keeps the family intersecting (as T was already in it) and does not increase any load.
- We may assume $\mathcal{T}$ is *maximal*: extend it to a maximal intersecting family and take its minimal sets. Allowing more sets (initially with share 0) can only decrease the optimum. The two properties above still hold for the new family: a singleton $\{c\}$ or a common element c would force c into every set of the original $\mathcal{T}$.

Certificates. To lower-bound the optimum for a fixed family $\mathcal{T}$, it suffices to give weights $p_1, \dots, p_k \ge 0$ with $\sum_i p_i = 1$ and $\sum_{i \in T} p_i \ge r$ for all $T \in \mathcal{T}$. Indeed,

$$\max_i \frac{|S_i|}{|U|} \;\ge\; \sum_i p_i \sum_{T \ni i} w_T \;=\; \sum_T w_T \sum_{i \in T} p_i \;\ge\; r.$$

Conversely, explicit shares show that the bound r is attained.

Classification. If all sets in $\mathcal{T}$ have size ≥ 3, the uniform weights $p_i = 1/k$ give ratio $\ge 3/k$. Otherwise, the 2-element sets in $\mathcal{T}$ pairwise intersect, so they form either a triangle $\{\{1,2\},\{1,3\},\{2,3\}\}$ or a star centered at 1. Up to permutation of $\{1, \dots, k\}$:

- $k \le 2$: The only set of size ≥ 2 is $\{1, 2\}$, which violates $\bigcap \mathcal{T} = \emptyset$. No valid $\mathcal{T}$ exists.
- **Triangle** $\{\{1,2\},\{1,3\},\{2,3\}\}$ (any $k \ge 3$): any further set would have to meet all three pairs, so it contains one of them, contradicting the antichain property. Weights $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots)$ give ratio $\ge 2/3$, attained by equal shares $\frac{1}{3}$.
- $k = 4$, star centered at 1 (not a triangle):
  - If the 2-element sets are $\{1, 2\}$ only, the other sets must meet $\{1, 2\}$ without containing it, so they lie in $\{\{1,3,4\},\{2,3,4\}\}$. Then $\{1, 3\}$ can be added, so this family is not maximal.

    - If they are $\{1,2\},\{1,3\}$, the only other possible set is $\{2,3,4\}$, and then $\{1,4\}$ can be added, so this is not maximal either.
    - This leaves **(4B)** $\mathcal{T}=\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$. Weights $p=(\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$ give ratio $\geq 3/5$, attained by shares $\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{2}{5}$.
- $k=5$: A similar (slightly longer) case analysis on the 2-element sets shows that, apart from the triangle, (4B) on 4 of the 5 indices, and the family of all 3-element sets, the only maximal families are the following three (this can also be confirmed by a short computer enumeration):
    - **(5A)** $\mathcal{T}=\{\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}$: weights $p=(\frac{2}{7},\frac{2}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7})$ give ratio $\geq 4/7$, attained by share $\frac{1}{7}$ for $\{1,2\}$ and $\frac{1}{7}$ for each of the other six sets.
    - **(5D)** $\mathcal{T}=\{\{1,2\},\{1,3\},\{1,4,5\},\{2,3,4\},\{2,3,5\}\}$: weights $p=(\frac{3}{9},\frac{2}{9},\frac{2}{9},\frac{1}{9},\frac{1}{9})$ give ratio $\geq 5/9$, attained by shares $\frac{1}{9},\frac{1}{9},\frac{3}{9},\frac{2}{9},\frac{2}{9}$.
    - **(5F)** $\mathcal{T}=\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3,4,5\}\}$: weights $p=(\frac{3}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7})$ give ratio $\geq 4/7$, attained by shares $\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{3}{7}$.

Therefore, the minimum achievable ratio $\max_i |S_i|/|U|$ for $k=3,4,5$ is $2/3$, $3/5$, and $5/9$, respectively. Note that the shares of (5F) on $|U|=7$ and of (5D) on $|U|=9$ are integral; they are exactly the set families used above for $K=4$ and $K=5$. Since we require $\max_i |S_i|/|U| \leq \frac{K}{2K-1}$:

- $k=3$ requires $\frac{2}{3}\leq\frac{K}{2K-1} \iff K\leq 2$;
- $k=4$ requires $\frac{3}{5}\leq\frac{K}{2K-1} \iff K\leq 3$;
- $k=5$ requires $\frac{5}{9}\leq\frac{K}{2K-1} \iff K\leq 5$.

A solution with fewer sets can always be padded with empty sets, so if k sets are impossible, so are fewer. This gives $f(2)\geq 3$, $f(3)\geq 4$, $f(4)\geq 5$, and $f(5)\geq 5$. For $K\geq 6$, we have $\frac{K}{2K-1}<\frac{5}{9}$, so $f(K)\geq 6$, which matches our $k=6$ construction.