Editorial - Tiling Game

Terminology

Every block occupies two consecutive rows and two consecutive columns. It is therefore useful to group the grid into $N \times M$ row pair/column pair sections.

For $0 \leq i < N$, call physical rows $2i$ and $2i + 1$ the i-th **row pair**. Similarly, for $0 \leq j < M$, call columns $2j$ and $2j + 1$ the j-th **column pair**.

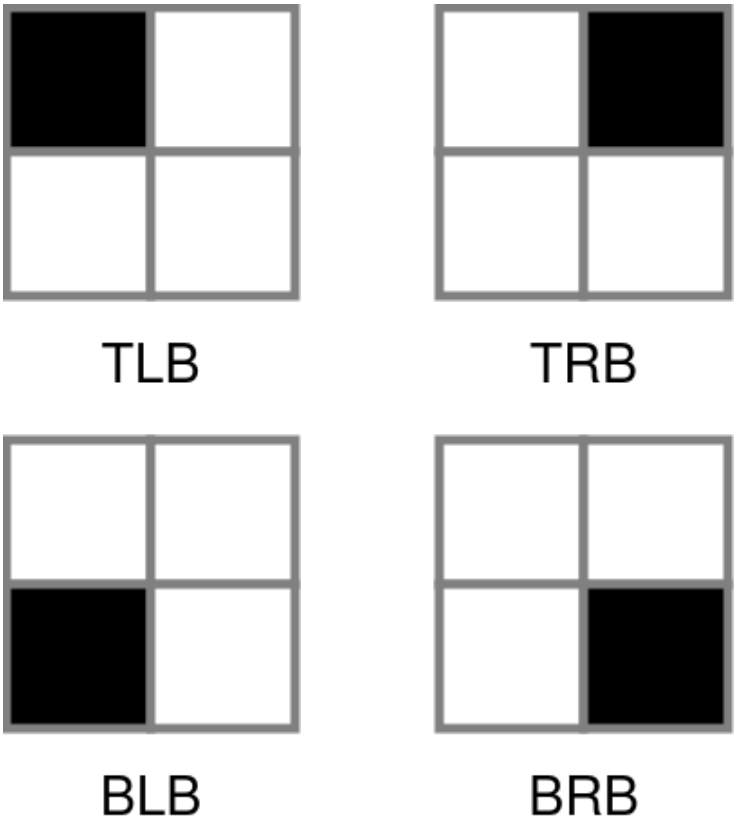
Within row pair i, row $2i$ is its **top row** and row $2i + 1$ is its **bottom row**. Within column pair j, column $2j$ is its **left column** and column $2j + 1$ is its **right column**.

We use the term **all-black** 2×2 **square** for a 2×2 subgrid whose four cells are black.

Subtask 1: $N = 2$ and $S = 1$

When $S = 1$, every block has exactly one black cell. We name the four possible block types according to the position of this black cell:

- `TLB`: black cell at the top-left;
- `TRB`: black cell at the top-right;
- `BLB`: black cell at the bottom-left;
- `BRB`: black cell at the bottom-right.



Construction

Place TLB and TRB blocks in the top row pair, and BLB and BRB blocks in the bottom row pair. Within each row pair, place the blocks from left to right until one of the row pairs becomes full.

As each block contains only one black cell, an all-black 2×2 square cannot be contained entirely within one row pair.

If the top row pair becomes full first, its bottom row remains entirely white. Similarly, if the bottom row pair becomes full first, its top row remains entirely white.

Therefore, after one row pair becomes full, placing the remaining blocks in the other row pair cannot create an all-black 2×2 square across the boundary between the two row pairs.

Hence the construction is valid.

Subtask 2

Here $N = M$ and N is even. We partition the $2N \times 2N$ grid into four equally sized square regions, each of size $N \times N$. Since N is even, each region has even dimensions and therefore consists of an integer number of 2×2 block positions.

Each block type occurs the same number of times, namely $\frac{N^2}{4}$. Therefore, we can assign one block type to each of the four square regions and fill each region using a fixed placement that contains no all-black 2×2 square.

For this construction, classify each block by the position of its unique white cell:

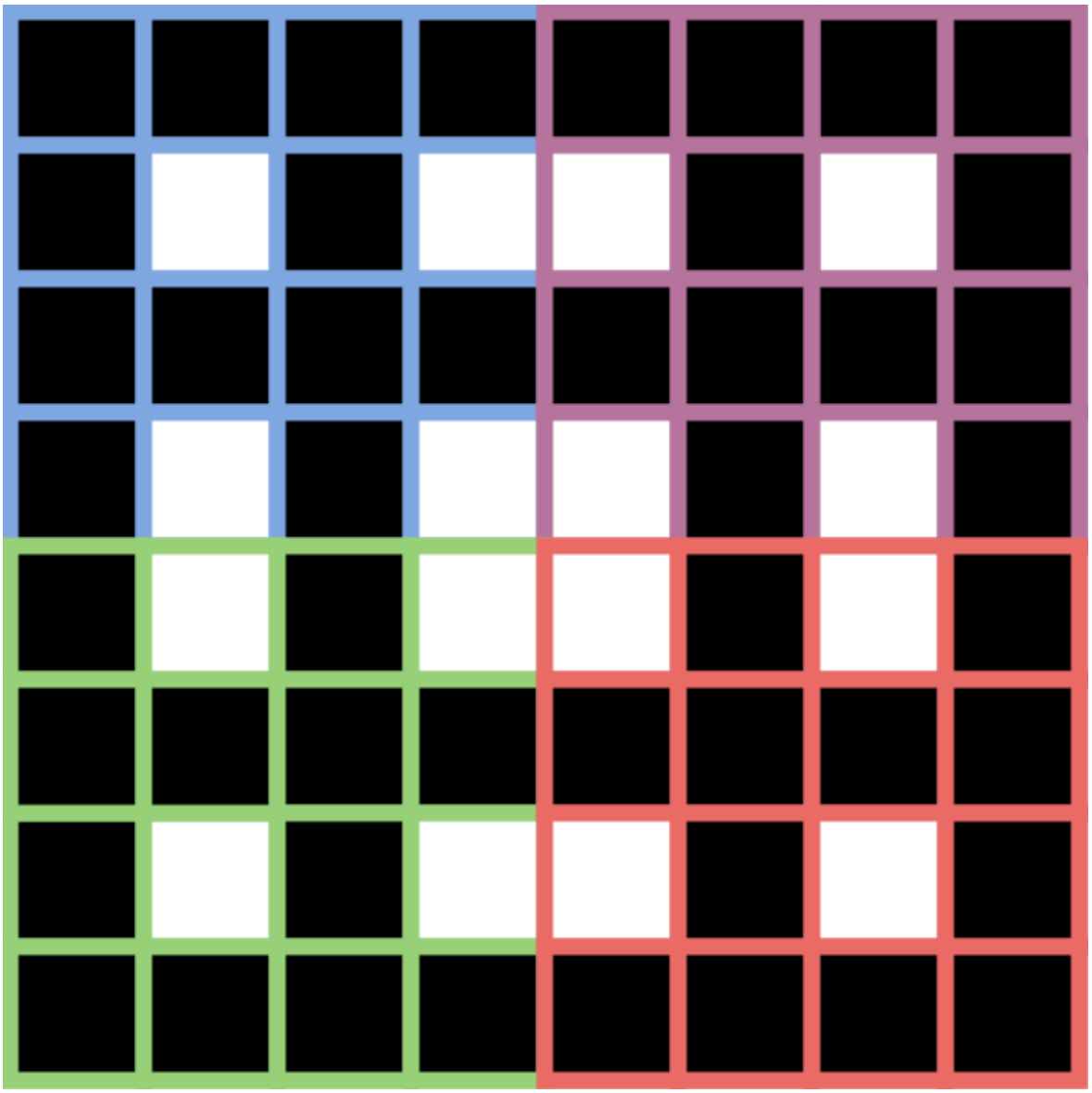
- BR: white cell at the bottom-right;
- BL: white cell at the bottom-left;
- TR: white cell at the top-right;
- TL: white cell at the top-left.

Assign the types as follows:

- BR blocks to the top-left region;
- BL blocks to the top-right region;
- TR blocks to the bottom-left region;
- TL blocks to the bottom-right region.

Each region receives exactly $N^2/4$ blocks, so all blocks can be placed.

For example, when $N = 4$, the resulting construction is shown below. Different outline colors indicate the four regions of the partition.



Within each region, every 2×2 block contains a white cell, so an all-black 2×2 square cannot be contained

in a single region. The arrangement of the four types also prevents an all-black square from crossing a boundary between two regions. Thus the construction is valid.

Subtask 3

We extend the construction from Subtask 1 to arbitrary N.

Construction

Maintain two active row pairs:

- the topmost row pair that still contains an empty cell;
- the bottommost row pair that still contains an empty cell.

Place TLB and TRB blocks in the current top row pair, and BLB and BRB blocks in the current bottom row pair, using the same left-to-right placement rule as in Subtask 1.

Whenever the top row pair becomes full, move the top pointer to the next row pair. Whenever the bottom row pair becomes full, move the bottom pointer to the previous row pair.

Correctness

As long as the active top and bottom row pairs are not adjacent, there is at least one complete row pair between them. Therefore, an all-black 2×2 square cannot contain cells from both active row pairs.

Within each active row pair, the argument from Subtask 1 applies: because all blocks assigned to that row pair have their unique black cells on the appropriate side, an all-black 2×2 square cannot be created inside the row pair.

Eventually, the top and bottom active row pairs may become adjacent. At that point, considering only these two row pairs gives exactly the situation handled in Subtask 1. Therefore no all-black 2×2 square is created.

Hence the construction is valid.

Subtask 5

Reduction to $S = 3$

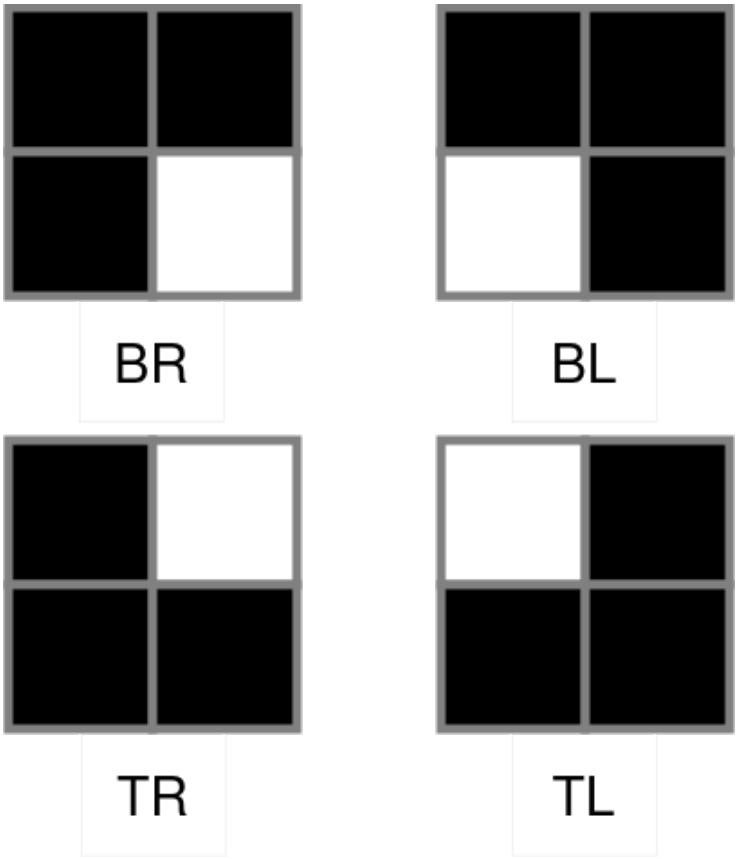
Some blocks may contain fewer than three black cells. Temporarily change some of their white cells to black until every block contains exactly three black cells. Record which cells were changed. Suppose we can place all these modified blocks without creating an all-black 2×2 square. After the construction is complete, change all recorded cells back to white.

Changing a black cell to white cannot create a new all-black 2×2 square. Therefore the resulting placement is also valid for the original blocks. Hence, it is sufficient to consider the case where **every block contains exactly three black cells**, or equivalently, exactly one white cell.

Block types

We classify a block according to the position of its unique white cell:

- BR: white cell at the bottom-right;
- BL: white cell at the bottom-left;
- TR: white cell at the top-right;
- TL: white cell at the top-left.



Construction

At every step, let:

- R_{top} be the topmost row pair containing an empty cell;
- R_{bottom} be the bottommost row pair containing an empty cell.

Place the block according to its type:

- BR: place it at the **leftmost** available position in R_{top};
- BL: place it at the **rightmost** available position in R_{top};
- TR: place it at the **leftmost** available position in R_{bottom};
- TL: place it at the **rightmost** available position in R_{bottom}.

The construction always places a block at a previously unused 2×2 block position. Since there are exactly NM block positions and exactly NM blocks, all blocks are eventually placed.

We now prove that this construction never creates an all-black 2×2 square.

Correctness

Assume, for contradiction, that at some point an all-black 2×2 square is created.

An all-black 2×2 square cannot be contained entirely inside one block, because every block contains one white cell. Consider the two rows and two columns of the all-black 2×2 square relative to the row pairs and column pairs.

Case 1: Both rows belong to the same row pair

The left two cells of the square belong to a block of type BL or TL, while the right two cells belong to a block of type BR or TR.

Suppose the left block was placed first. At that moment, the cell immediately to its right was empty. However, BL and TL blocks are always placed at the rightmost available position. Therefore the left block could not have been placed there.

Conversely, suppose the right block was placed first. At that moment, the cell immediately to its left was empty. However, BR and TR blocks are always placed at the leftmost available position. Again, this is impossible.

Thus an all-black 2×2 square cannot have both rows in the same row pair.

Case 2: Both columns belong to the same column pair

The top two cells belong to a block of type TR or TL, while the bottom two cells belong to a block of type BR or BL.

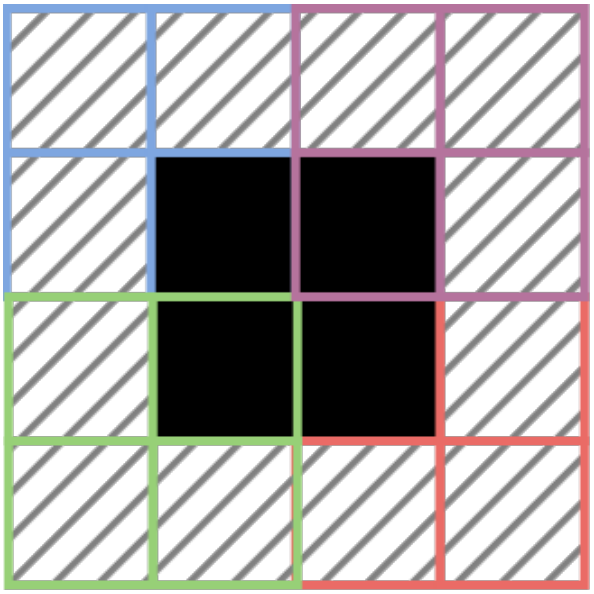
Suppose the top block was placed first. At that moment, the cell immediately below it was empty. However, TR and TL blocks are placed in the bottommost available row pair. Therefore the top block could not have been placed there.

Conversely, suppose the bottom block was placed first. At that moment, the cell immediately above it was empty. However, BR and BL blocks are placed in the topmost available row pair. This is also impossible.

Therefore an all-black 2×2 square cannot have both columns in the same column pair.

Case 3: The square crosses both a row-pair boundary and a column-pair boundary

It remains to consider an all-black 2×2 square whose two rows belong to different row pairs and whose two columns belong to different column pairs.



Let A, B, C, D denote the four blocks containing the four black cells, as shown in the figure with blocks represented by distinct outline color.

From the positions of the unique white cells, the possible types are:

- the top-left cell belongs to BL, TR, or TL;
- the top-right cell belongs to BR, TR, or TL;
- the bottom-left cell belongs to BR, BL, or TL;
- the bottom-right cell belongs to BR, BL, or TR.

We consider the type of the block containing the top-left cell.

Subcase 3.1: The top-left cell belongs to TR or TL

Such a block has its black cells on its bottom row, so it is placed in the bottommost available row pair.

Therefore, the blocks containing the bottom-left and bottom-right cells cannot be of type BR or BL. Otherwise, one of those blocks would have been placed in a row pair strictly below the row pair containing the TR/TL block, contradicting the topmost/bottommost placement rule.

Hence:

- the bottom-left cell must belong to a TL block;
- the bottom-right cell must belong to a TR block.

But TL blocks are always placed at the rightmost available position, whereas TR blocks are placed at the leftmost available position. Therefore the TL block cannot be to the left of the TR block.

This is a contradiction.

Subcase 3.2: The top-left cell belongs to BL

The top-right cell cannot belong to a BR block. Otherwise, both blocks would lie in the same row pair, with the BL block to the left of the BR block.

The BL block is always placed at the rightmost available position, while the BR block is always placed at the leftmost available position. Whichever of the two blocks was placed first would therefore contradict the placement rule.

Hence the top-right cell belongs to a TR or TL block.

We are then in the same situation as in Subcase 3.1: the bottom cells must belong to TL and TR blocks respectively, which requires a TL block to be placed to the left of a TR block. This is impossible because TL blocks are placed from the right and TR blocks from the left.

Thus this subcase also leads to a contradiction.

All possible cases lead to a contradiction. Therefore the construction never creates an all-black 2×2 square.

Hence the strategy is correct.